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CONFIDENCE INTERVALS AND SAMPLE SIZES IN THE MEASUREMENT OF SIGNAL AND NOISE POWERS, SIGNAL-TO-NOISE RATIOS AND PROBABILITY OF ERROR

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ABSTRACT

This report considers the estimation errors involved in both discrete and continuous estimates of certain parameters of a Gaussian random process. For discrete estimates, the confidence interval concept is used to obtain probabilistic bounds on the estimation errors. Roughly analogous results are also obtained for continuous estimates. The bounds obtained are useful for a) determining the accuracy of an estimate given the value of the estimate and the number of samples used (or for the continuous case the effective TW_s) and, b) for determining roughly the number of samples required (or the effective TW_s) to provide an estimate of a specified accuracy. The bounds are presented graphically and examples of their use are given.

A second result is the derivation of an approximate, but convenient and reasonably accurate method for evaluating the non-central t-distribution by means of tables of the normal distribution. This allows certain calculations to be made that are not now possible with existing tables.

I. INTRODUCTION

In many applications it is important to estimate certain parameters of a random process such as its mean or variance. For practical reasons these estimations must be based on finite data records and thus are subject to random errors. This report is concerned with the magnitude of these errors when the process may be considered to be stationary and Gaussian.

Two methods of parameter estimation will be considered in this analysis. First, estimates based on n independent samples of the process will be investigated. Probabilistic bounds on the estimation errors will be obtained through the use of the confidence interval concept. Secondly, estimates based on a continuous time average will be considered and corresponding probabilistic error bounds obtained.

II. DISCRETE ESTIMATION ERRORS

Denote the n samples on which the estimate is to be based by $x_1, x_2, -- x_n$, the true value of the parameter to be estimated by Θ, and the estimate of Θ by $\hat{\Theta} = \hat{\Theta}(x_1, x_2, -- x_n)$. Assume that the only unknown parameter in the known distribution of the independent, identically distributed x_i is Θ and denote the corresponding density function by $p_1(x;\Theta)$. Then the density function for $\hat{\Theta}, p_2(\hat{\Theta};\Theta)$, may be determined as a function of the unknown parameter Θ. The question then arises as to whether or not $p_2(\hat{\Theta};\Theta)$ can be used to make a probabilistic statement concerning the value of Θ given a value of $\hat{\Theta}$. The answer to this question involves two fundamentally different concepts depending upon the nature of Θ in the process being sampled. These concepts are discussed in the following paragraphs.

Consider first a situation in which a large number of random processes are available each having a different value of Θ. An estimate of Θ is to be obtained by first randomly selecting a particular process and then taking n independent samples of a sample function in this process. After calculating $\hat{\Theta}$ from these samples it is desired to make a statement of the type $P_r \{ a < \Theta < b \} = 1 - \epsilon$, where $0 < \epsilon < 1$). For this situation such a statement is readily obtained by considering Θ to have a known density function $p(\Theta)$, noting that $p_2(\hat{\Theta};\Theta)$ is the conditional density of $\hat{\Theta}$ given Θ, and applying Bayes' theorem [1] to obtain

$$P_r\{a < \Theta < b\} = \int_a^b p(\Theta/\hat{\Theta})\, d\Theta$$

where

$$p(\Theta/\hat{\Theta}) = \frac{p_2(\hat{\Theta};\Theta)\, p(\Theta)}{\int_{-\infty}^{\infty} p_2(\hat{\Theta};\Theta)\, p(\Theta)\, d\Theta}$$

Note that this is just the conditional probability of $a < \Theta < b$ given a value of $\hat{\Theta}$.

In general $p_2(\hat{\Theta};\Theta)$ will involve n, the number of samples, in such a way that with $a < \hat{\Theta}$ and $b > \hat{\Theta}$

$$\lim_{n \to \infty} P_r\{a < \Theta < b\} = 1$$

while simultaneously $b - a$ is approaching zero. Thus, this approach allows a probabilistic statement to be made concerning the error in estimating Θ by $\hat{\Theta}$ when n is specified and conversely allows n to be determined to give a specified error.

Usually, however, estimates are to be made using samples taken from a sample function of a <u>single</u> process in which the parameter Θ is an unknown constant. In this case $p(\Theta) = \delta(\Theta - c)$ (where c is the unknown constant value of Θ and $\delta(x)$ is the unit dirac delta function) and the above method leads to the meaningless result

$$P_r\{a < \Theta < b\} = \begin{cases} 1 \text{ if } a < \Theta < b \\ 0 \text{ if } \Theta < a \text{ or } \Theta > b \end{cases}$$

Consideration of this problem has led to the development of the theory of confidence intervals by J. Neyman[1]. The following paragraphs discuss this theory as it applies to the analysis of this report.

In the density function $p_2(\hat{\Theta};\Theta)$ defined above substitute an arbitrary number, say Θ', for the unknown parameter Θ. $p_2(\hat{\Theta};\Theta')$ is then completely defined and probability statements of the type

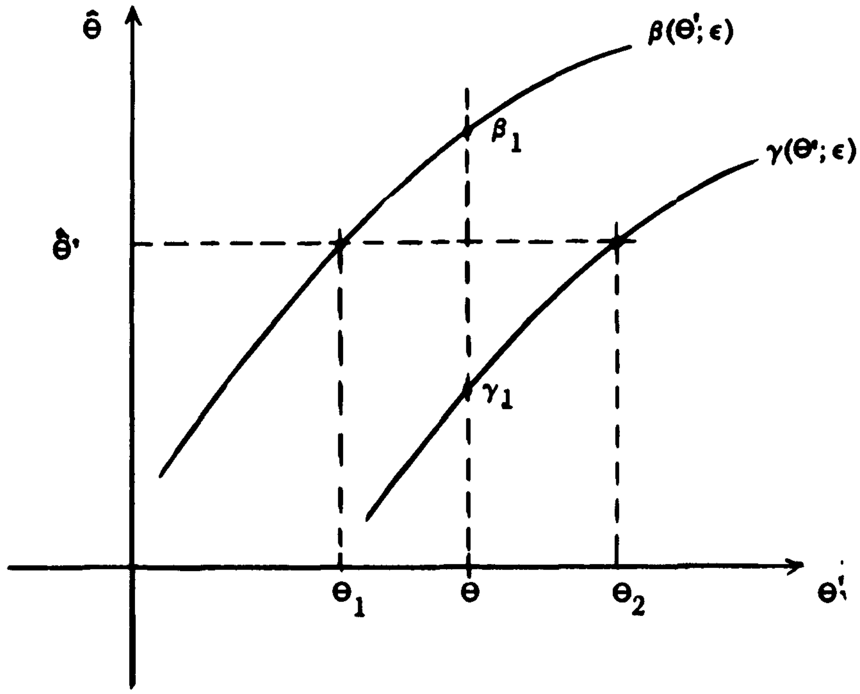
$$P_r\{\gamma < \hat{\Theta} < \beta\} = \int_{\gamma}^{\beta} p_2(\hat{\Theta};\Theta')\,d\hat{\Theta} = 1-\epsilon \tag{1}$$

with

$$\int_{-\infty}^{\gamma} p_2(\hat{\Theta};\Theta')\,d\hat{\Theta} = \epsilon_1$$

$$\int_{\beta}^{\infty} p_2(\hat{\Theta};\Theta')\,d\hat{\Theta} = \epsilon - \epsilon_1$$

may be inferred for any value of Θ'. Now, as Θ' and/or ϵ is varied in these expressions γ and β will vary. Thus $\gamma = \gamma(\Theta', \epsilon)$ and $\beta = \beta(\Theta', \epsilon)$. Assuming γ and β to be monotonic functions of Θ' these may be plotted in the $\hat{\Theta}$, Θ' plane as shown below.



With $\Theta' = \Theta$, the true value of the parameter to be estimated, and $\hat{\Theta} = \hat{\Theta}'$, a value of $\hat{\Theta}$ determined from $\hat{\Theta}(x_1 \,\text{--}\, x_n)$ and n specific sample values, consider the relations

$$(a)\ \gamma_1 < \hat{\Theta}' < \beta_1 ; \qquad (b)\ \Theta_1 < \Theta < \Theta_2$$

where $\gamma_1, \gamma_2, \Theta_1, \Theta_2$ are defined by the values of Θ and $\hat{\Theta}'$ as indicated above. Now, for any particular Θ, the above plot shows that a value of $\hat{\Theta}'$ satisfying relation (a) also satisfies relation (b). Conversely, if Θ' satisfies (b) then (a) is satisfied. From this the important conclusion is drawn that the statement $\gamma_1 < \hat{\Theta}' < \beta_1$ is completely equivalent to $\Theta_1 < \Theta < \Theta_2$ and thus that

$$P_r \{ \gamma_1 < \hat{\Theta}' < \beta_1 \} = P_r \{ \Theta_1 < \Theta < \Theta_2 \} = 1 - \epsilon \tag{2}$$

It is in the interpretation of this statement that the difference between this approach and that using Bayes' theorem becomes apparent.

From the above graph it is observed that

$$\hat{\Theta}' = \beta(\Theta_1, \epsilon) = \gamma(\Theta_2, \epsilon)$$

Thus Θ_1 and Θ_2 are random variables whose value depends upon the values of the n samples used to obtain the estimate $\hat{\Theta}'$. In words the above probability statement thus becomes "the probability that the random <u>interval</u> $[\Theta_1, \Theta_2]$ will cover true parameter value Θ is $1 - \epsilon$". This is to be contrasted to the Bayes' theorem statement which reads " the probability that the random <u>variable</u> Θ lies in the fixed interval $[a,b]$ is $1 - \epsilon$."

In general, Θ_1 and Θ_2 will be functions of n in such a manner that

$$\lim_{n \to \infty} P_r \{ \Theta_1 < \Theta < \Theta_2 \} = 1$$

Thus, for a given ϵ and n it will be possible to make a probabilistic statement concerning the estimation error by the <u>a priori</u> statement: "before the n samples are taken the probability that the interval constructed using these samples, (i.e. $[\Theta_1, \Theta_2]$ where $\hat{\Theta}' = \beta(\Theta_1, \epsilon) = \gamma(\Theta_2, \epsilon)$) covers the true parameter value Θ is $1 - \epsilon$."

Since the values Θ_1 and Θ_2 are random variables the interval will also be a random variable from one set of samples to the next. However, when n is sufficiently large to give useful estimates, the length of the interval varies by a negligible amount from sample to sample. Because of this it is possible to obtain an accurate indication of the magnitude of the estimation error that will be exceeded only 100 ϵ % of the time.

The interval $[\Theta_1, \Theta_2]$ is called the <u>confidence interval</u> for the parameter Θ, Θ_1 and Θ_2 are called the <u>confidence limits</u>, and $1 - \epsilon$ the <u>confidence level</u>. The following paragraph describes a general method for determining the confidence limits and thus the confidence interval, for a given confidence level.

From above, $\gamma(\Theta', \epsilon)$ is defined by

$$\int_{-\infty}^{\gamma} p_2(\hat{\Theta};\Theta')\, d\hat{\Theta} = \epsilon_1$$

where ϵ, and $p_2(\hat{\Theta};\Theta')$ are specified. Also, from the graph Θ_2 is the value of Θ' in $\gamma(\Theta', \epsilon)$ which makes $\gamma = \hat{\Theta}'$. Thus, the upper confidence limit Θ_2 is the value of Θ' in $p_2(\hat{\Theta};\Theta')$ such that

$$\int_{-\infty}^{\hat{\Theta}'} p_2(\hat{\Theta};\Theta')\, d\hat{\Theta} = \epsilon_1 \tag{3}$$

In an analogous manner the lower confidence limit Θ_1 is the value of Θ in $p_2(\hat{\Theta};\Theta')$ such that

$$\int_{\hat{\Theta}'}^{\infty} p_2(\hat{\Theta};\Theta')\, d\hat{\Theta} = \epsilon - \epsilon_1 \tag{4}$$

where $1 - \epsilon$ is the desired confidence level, typically 0.95 or 0.99. Clearly, the length of the confidence interval, $\Theta_2 - \Theta_1$, will depend upon ϵ_1 and will have at least one minimum for $0 < \epsilon_1 < \epsilon$. In general, however, it is quite difficult to determine this extremum and the usual practice is to take $\epsilon_1 = \epsilon/2$ which is the optimum value when $p_2(\hat{\Theta};\Theta')$ is symmetric in $\hat{\Theta}$ and Θ'. Following this procedure Θ_1 and Θ_2 are given by the implicit relations

$$\int_{-\infty}^{\hat{\theta}'} p_2(\hat{\theta};\theta_2)\,d\hat{\theta} = \epsilon/2 \qquad (5)$$

or in more convenient notation

$$P_r\{\hat{\theta} < \hat{\theta}' \mid \theta_2\} = \epsilon/2 \qquad (5a)$$

and

$$\int_{\hat{\theta}'}^{\infty} p_2(\hat{\theta};\theta_1)\,d\hat{\theta} = \epsilon/2 \qquad (6)$$

or

$$P_r\{\hat{\theta} > \hat{\theta}' \mid \theta_1\} = \epsilon/2 \qquad (6a)$$

Note that Eqs. (5a) and (6a) are not conditional probabilities but are simply shorthand statements for the previous equations.

1. Confidence Intervals for Probability of Error

In this section confidence intervals for the estimation of probability of error will be derived by considering the n samples to be drawn independently from a discrete random variable having the binomial distribution. In all of the remaining sections the samples will be assumed to be normally distributed.

Consider the occurence of errors in a digital receiver to be independent from baud to baud and define a random variable x by

$x = 1$ if an error occurred in a given baud,

$x = 0$ if no error occurred in a given baud.

The following relations are then readily determined.

$$P(x) = (Pe)^x (1 - Pe)^{1-x} \qquad x = 0,\ 1 \qquad (7)$$

and

$$P(\hat{P}e) = \binom{n}{n\hat{P}e} (Pe)^{n\hat{P}e} (1 - Pe)^{n(1-\hat{P}e)} \quad \hat{P}e = \frac{0}{n}, \frac{1}{n}, \dots \frac{n}{n} \tag{8}$$

where

$$\binom{n}{n\hat{P}e} = \frac{n!}{(n\hat{P}e)!\,(n-n\hat{P}e)!}$$

Pe = true probability of error

$\hat{P}e$ = estimate of Pe

$$= \frac{1}{n} \sum_{i=1}^{n} x_i = \frac{\text{number of errors in n bauds}}{n}$$

The x_i's are n independent samples of the random variable x of Eq. (7).

Analogous to Eq. (5), [2] the upper confidence limit Pe_2 is the value of Pe such that

$$\sum_{\hat{P}e=0}^{\hat{P}e} \binom{n}{n\hat{P}e} (Pe)^{n\hat{P}e} (1 - Pe)^{n(1-\hat{P}e)} \leq \frac{\epsilon}{2} \tag{9}$$

Similarly the lower confidence limit Pe_1 is the value of Pe such that

$$\sum_{\hat{P}e=\hat{P}'e}^{1} \binom{n}{n\hat{P}e} (Pe)^{n\hat{P}e} (1 - Pe)^{n(1 - \hat{P}e)} \leq \frac{\epsilon}{2} \tag{10}$$

Reference [4] gives confidence intervals determined from Eqs. (9) and (10) that are useful for $\hat{P}e > 0.05$ and for n = 10, 15, 20, 30, 50, 100, 250, and 1000. Usually, however, the values of $\hat{P}e$ of interest are less than 0.05. Since tables of the binomial distribution are not available for $Pe < 0.01$ and $n > 1000$ the normal approximation to the binomial distribution has been used. With $0 \leq a < b \leq 1$ this approximation is [5]

$$\sum_{\hat{P}e=a}^{b} \binom{n}{n\,\hat{P}e} (Pe)^{n\,\hat{P}e} (1 - Pe)^{n(1 - \hat{P}e)} \simeq \frac{1}{\sqrt{2\pi}} \int_{L_1}^{U_1} e^{-1/2t^2} dt \qquad (11)$$

where

$$U_1 = \frac{a - Pe}{[Pe(1 - Pe)/n]^{1/2}}$$

$$L_1 = \frac{b - Pe}{[Pe(1 - Pe)/n]^{1/2}}$$

The error in Eq. (11) is such that the normal approximation is small by a negligible amount when $n\,Pe\,(1 - Pe) > 25$. Thus from Eq. (9) Pe_2 must be such that

$$\frac{\epsilon}{2} = \frac{1}{\sqrt{2\pi}} \int_{L_2}^{U_2} e^{-1/2t^2} dt \qquad (12)$$

where

$$U_2 = \frac{\hat{P}'e - Pe_2}{[Pe_2(1 - Pe_2)/n]^{1/2}}$$

$$L_2 = \frac{- Pe_2}{[Pe_2(1 - Pe_2)/n]^{1/2}}$$

Under the conditions $n\,Pe_2(1 - Pe_2) > 25$ and $Pe_2 < 0.1$

$$L_2 \simeq -\sqrt{n\,Pe_2} < -5$$

Thus with these restrictions on n and Pe_2, Eq. (12) can be approximated by

$$\frac{\epsilon}{2} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{U_2} e^{-1/2t^2} dt \qquad (13)$$

Introducing similar approximations into Eq. (10) shows that Pe_1 must be such that

$$\frac{\epsilon}{2} = \frac{1}{\sqrt{2\pi}} \int_{L_3}^{\infty} e^{-1/2t^2}\, dt \qquad (14)$$

where

$$L_3 = \frac{\hat{P}'e - Pe_1}{[Pe_1(1 - Pe_1)/n]^{1/2}}$$

With K_ϵ = defined by

$$\frac{\epsilon}{2} = \frac{1}{\sqrt{2\pi}} \int_{K_\epsilon}^{\infty} e^{-1/2t^2}\, dt \qquad (15)$$

These relations for Pe_2 and Pe_1 become

$$\frac{\hat{P}'e - Pe_2}{[Pe_2\,(1-Pe_2)/n]^{1/2}} = -K_\epsilon$$

$$\frac{\hat{P}'e - Pe_1}{[Pe_1\,(1-Pe_1)/n]^{1/2}} = K_\epsilon$$

Using again the conditions $n\,Pe\,(1-Pe) > 25$ and $Pe < 0.1$ yields the final results

$$Pe_1 = \hat{P}e' \;[1 - K_\epsilon \times 1/\sqrt{n\,\hat{P}e'}]$$

$$Pe_2 = \hat{P}e' \;[1 + K_\epsilon \times 1/\sqrt{n\,\hat{P}e'}] \qquad (16)$$

These results are plotted in Figures 1 and 2 for $\epsilon = 0.05$ and 0.01 respectively and examples of their use are given in Section II-5.

2. Confidence Intervals for Noise Power

The estimate of the noise power N_o is given by

$$\hat{N}_o \equiv \hat{\sigma}^2 = \frac{1}{n-1} \sum_{i=1}^{n} (x_i - \hat{\mu})^2 \tag{17}$$

where

$$\hat{\mu} = \frac{1}{n} \sum_{i=1}^{n} x_i$$

It is known [2] that the quantity $\frac{\hat{\sigma}^2}{\sigma^2}(n-1)$ is distributed by the chi-square distribution with $(n-1)$ degrees of freedom i.e.,

$$p(\chi^2) = \frac{1}{\Gamma(\frac{n-1}{2})} (2)^{-(n-1)/2} (\chi^2)^{(n-3)/2} e^{-(1/2)\chi^2} \tag{18}$$

where $\chi^2 \geq 0$

$$\chi^2 = (n-1) \frac{\hat{\sigma}^2}{\sigma^2} = (n-1) \frac{\hat{N}_o}{N_o}$$

Now, from Eq. (5) the upper confidence limit N_{o2} is the value of N_o such that

$$\int_0^{(n-1)\hat{N}'_o/N_o} p(\chi^2)\, d\chi^2 = \frac{\epsilon}{2} \tag{19}$$

In [6] it is shown that the distribution for χ is approximately Gaussian with mean $n-1$ and variance $\frac{1}{2}$. for n as small as 5. Thus Eq. (19) can be approximated by

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{U_4} e^{-1/2t^2}\, dt = \frac{\epsilon}{2} \tag{20}$$

where

$$U_4 = \frac{\sqrt{n-1}\,[\sqrt{\hat{N}_o'/N_o} - 1]}{1/\sqrt{2}}$$

With K_ϵ as in Eq. (15) the final expression for N_{o2} is thus

$$N_{o2} = \hat{N}_o' \left[1 - \frac{K_\epsilon}{\sqrt{2(n-1)}} \right]^{-2} \tag{21}$$

Similarly the lower confidence limit is found to be

$$N_{o1} = \hat{N}_o' \left[1 + \frac{K_\epsilon}{\sqrt{2(n-1)}} \right]^{-2} \tag{22}$$

For $n > 30$, the error in these confidence limits due to the approximation of Eq. (20) is less than 1%. Figure 3 presents these results graphically.

3. Confidence Intervals for Signal Power

The maximum-likelihood estimate of the signal power S_o is [1]

$$\hat{S}_o \equiv (\hat{\mu})^2 = \left[\frac{1}{n} \sum_{i=1}^{n} x_i \right]^2 .$$

From Eq. (6a) the lower confidence limit S_{o1} must be such that

$$P_r \{ \hat{S}_o < \hat{S}_o' \mid S_{o1} \} = 1 - \epsilon/2 \tag{23}$$

Thus since $\hat{\mu}$ can be negative, Eq. (23) becomes

$$P_r \{ -\sqrt{\hat{S}_o'} < \hat{\mu} < \sqrt{\hat{S}_o'} \mid S_{o1} \} = 1 - \epsilon/2 \tag{24}$$

It is known [2] that

$$t_c = \frac{\hat{\mu} - \mu}{\sigma / \sqrt{n}}$$

is distributed by the t-distribution with n - 1 degrees of freedom. Thus S_{ol} is the square of the value of μ such that

$$1 - \frac{\epsilon}{2} = \int_{-L_5}^{U_5} p(t, n-1)\, dt \qquad (25)$$

where

$$U_5 = \frac{\sqrt{\hat{S}'_o} - \mu}{\hat{\sigma}'/\sqrt{n}}$$

$$L_5 = \frac{\sqrt{\hat{S}'_o} + \mu}{\hat{\sigma}'/\sqrt{n}}$$

$p(t, n-1)$ = the central t-distribution with n - 1 degrees of freedom

To simplify this implicit relation for μ rewrite Eq. (25) as

$$\int_{-\infty}^{U_5} p(t, n-1)\, dt + \int_{-\infty}^{L_5} p(t, n-1)\, dt = 2 - \epsilon/2 \qquad (26)$$

Since this expression is even in μ, $\mu \geq 0$ are the only values of interest.

Requiring $\sqrt{n \hat{S}'_o} / \hat{\sigma}' > 5$ and $n > 10$ makes the second integral in Eq. (26) equal unity to four decimals. Under these conditions Eq. (26) can be replaced by

$$\int_{-\infty}^{U_5} p(t, n-1)\, dt = 1 - \epsilon/2 \qquad (27)$$

Defining $t_\epsilon(n)$ by

$$\int_{-\infty}^{t_\epsilon(n)} p(t,\ n-1)\, dt = 1-\epsilon/2 \qquad (28)$$

gives for the lower confidence limit

$$S_{o1} = \hat{S}'_o \left[1 - \frac{t_\epsilon(n)}{\sqrt{n\hat{\rho}'}}\right]^2 \qquad (29)$$

A similar procedure gives

$$S_{o2} = \hat{S}'_o \left[1 + \frac{t\epsilon\,(n)}{\sqrt{n\hat{\rho}'}}\right]^2 \qquad (30)$$

where

$$\hat{\rho}' = [\hat{\mu}'/\hat{\sigma}']^2 = \hat{S}'_o/\hat{N}'_o$$

For n > 120 the t-distribution is closely approximated by the normal distribution with mean zero and variance unity. Thus when n > 120, $t_\epsilon(n) = K_\epsilon$ may be used in Eqs. (29) and (30). Figures 4 and 5 illustrate these results.

4. Confidence Intervals for Signal-to-Noise Ratio

The estimate of the SNR, ρ, is defined by

$$\hat{\rho} = [\hat{u}/\hat{\sigma}]^2 = \hat{S}_o/\hat{N}_o$$

Thus from Eq. (6) the lower confidence limit ρ_1 must satisfy

$$P_r\ \{\hat{\rho} < \hat{\rho}' \mid \rho_1\} = 1 - \epsilon/2 \qquad (31)$$

or, since $\hat{\mu}$ can be negative,

$$P_r\ \{-\sqrt{\hat{\rho}'} < \frac{\hat{\mu}}{\hat{\sigma}} < \sqrt{\hat{\rho}'} \mid \rho_1\} = 1 - \epsilon/2 \qquad (32)$$

Now, the quantity

$$t = \frac{z + \delta}{\sqrt{w}} = \sqrt{n}\,\frac{\hat{\mu}}{\hat{\sigma}}$$

where

$$z = \frac{\hat{\mu} - \mu}{\sigma/\sqrt{n}}$$

$$\delta = \frac{\mu\sqrt{n}}{\sigma} = \sqrt{\rho\, n}$$

$$w = \frac{\hat{\sigma}^2}{\sigma^2}$$

is known [7] [8] to be distributed by the non-central t-distribution with n - 1 degrees of freedom. Thus ρ_1 is found by determining a value of δ such that

$$\int_{-\sqrt{\hat{\rho}' n}}^{\sqrt{\hat{\rho}' n}} p(t, \delta, n - 1)\, dt = 1 - \epsilon/2 \tag{33}$$

where p (t, δ, n - 1) is the non-central t-distribution with n - 1 degrees of freedom. Unfortunately, tables of the non-central t-distribution are not suited for calculations of this type since trial and error determination of δ is involved. This difficulty can be circumvented in the following manner.

With the above definitions of t, δ, and w, and defining δ_1 to be the value of δ satisfying Eq. (33), Eq. (32) can be rewritten as

$$P_r \left\{ -\sqrt{n\,\hat{\rho}'} < \frac{z + \delta_1}{\sqrt{w}} < \sqrt{n\,\hat{\rho}'} \right\} = 1 - \epsilon/2 \tag{34}$$

or

$$P_1 + P_2 = 1 - \epsilon/2 \tag{35}$$

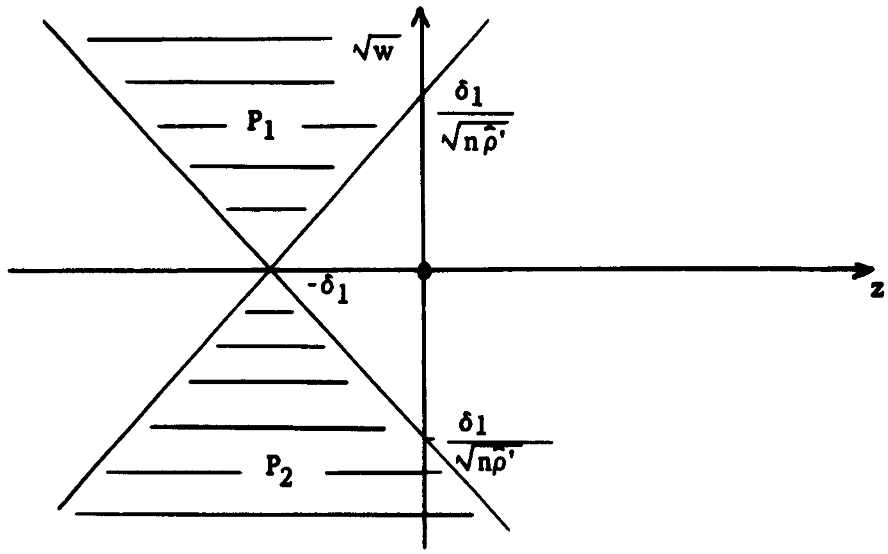
where

$$P_1 = P_r\{ -z - \sqrt{n\hat{\rho}'}\,\sqrt{w} < \delta_1 < -z + \sqrt{n\hat{\rho}'}\,\sqrt{w} \mid \sqrt{w} > 0 \}\, P_r\{ \sqrt{w} > 0 \}$$

$$P_2 = P_r\{ -z + \sqrt{n\hat{\rho}'}\,\sqrt{w} < \delta_1 < -z - \sqrt{n\hat{\rho}'}\,\sqrt{w} \mid \sqrt{w} < 0 \}\, P_r\{ \sqrt{w} > 0 \}$$

(At this point it is convenient to ignore the fact that $P_r\{\sqrt{w} < 0\} = 0$, so that the following discussion may be simplified. This fact will be reintroduced at a later point.)

Now, using the joint density function for z and $\sqrt{w}$, P_1 and P_2 can be found by integrating this over the regions of the z, $\sqrt{w}$ plane indicated below.



However, the difficulty in evaluating this double integral for general n makes it convenient to determine these probabilities in the following manner. Let

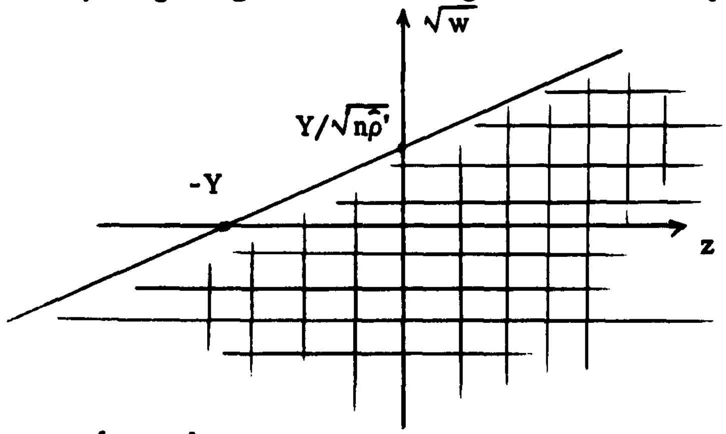
$$x \equiv -z - \sqrt{n\hat{\rho}'}\,\sqrt{w}$$

$$y \equiv -z + \sqrt{n\hat{\rho}'}\,\sqrt{w}$$

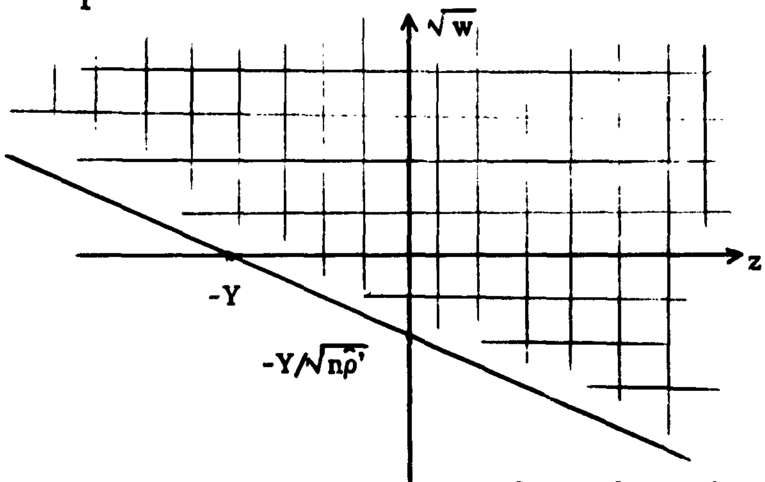
It is known [2] that z is normally distributed with mean zero and variance unity, and that (from Sect. II-2) $\sqrt{w}$ is approximately normally distributed with mean unity and variance $[2(n-1)]^{-1}$. It is also known that the normal approximation

for the distribution of $\sqrt{w}$ is quite accurate even for small n and that z and $\sqrt{w}$ are statistically independent [2]. Therefore the distribution for -x and y is even more accurately approximated by the normal distribution with mean $\sqrt{n\hat{\rho}'}$ and variance $1 + n\hat{\rho}' \; [2(n-1)]^{-1}$. Thus, this approximation allows probabilities such as $P_r\{x < X\}$ or $P_r\{y < Y\}$ to be accurately determined using tables of the normal distribution. It will now be shown that $P_r\{x < \delta_1\} - P_r\{y < \delta_1\} = P_1 - P_2$, when P_1 and P_2 are as previously defined, and thus that the normal distribution may be used to evaluate Eq. (33).

Thinking again in terms of the joint density function for t and $\sqrt{w'}$, $P_r\{y < Y\}$ would be obtained by integrating this over the region in the t, $\sqrt{w'}$ plane below.



Similarly $P_r\{x < Y\}$ would be obtained by integration over the following region.



From this, with $Y = \delta_1$, it follows directly that $P_r\{x < Y\} - P_r\{y < Y\} = P_1 - P_2$ which was to be proved. Now, from Eq. (35) $P_2 \leq P_r\{\sqrt{w} < 0\}$. However, it is

known that $P_r\{\sqrt{w} < 0\} = 0$. Therefore if the normal approximation to the distribution of $\sqrt{w}$ is to be accurate $P\{\sqrt{w} < 0\}$ calculated from it must be negligibly small. Requiring $[2(n-1)]^{-1/2} < 1/4$, or equivalently $n > 9$, makes $P_r\{\sqrt{w} < 0\}$ and thus P_2 less than 10^{-4}. With this restriction on n, Eq. (35) becomes, with negligible error,

$$P_r\{x < \delta_1\} - P_r\{y < \delta_1\} = 1 - \epsilon/2 \tag{36}$$

Introducing the normal distribution and recalling that $\delta_1^2 = n\rho_1$, Eq. (36) is evaluated as

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{U_6} e^{-1/2t^2}\, dt - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{U_6} e^{-1/2t^2}\, dt = 1 - \epsilon/2 \tag{37}$$

where

$$U_6 = \frac{\sqrt{n\rho_1} + \sqrt{n\hat{\rho}'}}{\{1+n\hat{\rho}'\,[2(n-1)]^{-1}\}^{1/2}}$$

$$U_7 = \frac{\sqrt{n\rho_1} - \sqrt{n\hat{\rho}'}}{\{1+n\hat{\rho}'\,[2(n-1)]^{-1}\}^{1/2}}$$

Since $\sqrt{n\rho_1} \geq 0$ the requirement

$$\frac{\sqrt{n\rho'}}{\{1+\rho'\,[2(n-1)]^{-1}\}^{1/2}} > 4$$

or approximately

$$n > 8\,[1 + \frac{2}{\hat{\rho}'}]$$

makes the first integral in Eq. (37) equal unity to 4 decimals. Therefore, under the double restriction

$$n > 9$$

$$n > 8\,[1 + \frac{2}{\hat{\rho}'}]$$

the lower confidence limit ρ_1 is found from

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{U_7} e^{-1/2\, t^2}\, dt = \frac{\epsilon}{2} \tag{38}$$

In terms of K_ϵ as previously defined the solution is thus

$$K_\epsilon = \frac{\sqrt{n\hat{\rho}'} - \sqrt{n\rho_1}}{\sqrt{1+n\hat{\rho}'\,[2(n-1)]^{-1}}}$$

or

$$\rho_1 = \hat{\rho}' \left[1 - \frac{K_\epsilon}{\sqrt{n}} \sqrt{\frac{1}{\hat{\rho}'} + \frac{n}{2(n-1)}}\right]^2 \tag{39}$$

A similar procedure shows that

$$\rho_2 = \hat{\rho}' \left[1 + \frac{K_\epsilon}{\sqrt{n}} \sqrt{\frac{1}{\hat{\rho}'} + \frac{n}{2(n-1)}}\right]^2 \tag{40}$$

Figures (6) through (9) present these results graphically.

5. Examples

a) The output of a digital receiver is observed for 6.0×10^4 bauds, and 24 errors are observed.

$$n = 6.0 \times 10^4$$

$$\hat{P}e = 4 \times 10^{-4}$$

From Figure 1, $2.4 \times 10^{-4} < Pe < 5.6 \times 10^{-4}$ with a confidence level of 95%.

b) It is desired to measure a probability of error which is expected to be about 10^{-3} with a confidence level of 95% and a confidence interval of $\pm$ 10% of the measured value. How many samples should be used? From Fig. 1 a measured value of

$Pe = 10^{-3}$ with $n = 4 \times 10^5$ samples would have the desired confidence level and interval. Therefore an approximate sample size of 4×10^5 should be used. Note, however, that the actual confidence interval must be determined after Pe has been measured.

c) A sample of n = 1000 gives

$$\hat{S}'_o = 1.0$$

$$\hat{N}'_o = 4.0$$

$$\hat{\rho}' = 0.25$$

From Figs. 4 , 3, and 6 respectively the 95% confidence intervals are

$$0.77 \leq S_o \leq 1.26$$

$$3.68 \leq N_o \leq 4.4$$

$$0.21 \leq \rho \leq 0.33$$

d) It is desired to measure an unknown noise power to within approximately $\pm$ 10% at a 99% confidence level. From Fig. 5 a sample size of 1.7×10^3 gives a confidence interval of +10%, - 9% for any value of $\hat{N}'_o$.

e) As a less trivial example of the use of confidence intervals, consider the problem of evaluating a low data rate binary communication system having the following theoretical performance characteristics.

$$P_e = \frac{1}{2} e^{-\beta} \qquad \beta \geq 0 \tag{41}$$

$$SNR = \beta \left[1 + 1/2\beta \right]^{-1} \qquad \beta \geq 0 \tag{42}$$

$$\text{where } \beta = \frac{E}{N_o} = \frac{\text{Signal energy per baud at receiver input}}{\text{Noise spectral density at receiver input}}$$

The system is to be evaluated by measuring either P_e or SNR and calculating an "effective" β from Eq. (41) or (42) respectively. Comparison of the calculated β with the value actually used in the test gives an indication of the departure of the system from ideal operation. The question arises as to which of the two measurement methods provides the desired measurement accuracy with the least number of samples. The following analysis demonstrates that for small β the two methods are approximately equivalent while for large β the number of samples required for SNR measurements is less, by several orders of magnitude, than the number required for P_e measurements.

Assume that the probability of an error is independent from baud to baud, that the noise component of the samples used in calculating SNR is independent from baud to baud, and normally distributed, and that the desired accuracy for the calculated "effective" β ($\hat{\beta}$) is $\pm$ 100α% at a confidence level of 100(1-ϵ)%. Then, Eq. (16) shows that the number of samples used for P_e measurements must be at least as large as the largest value of n satisfying the relation

$$\hat{P}'_e \left[1 \pm K_\epsilon / \sqrt{n \hat{P}'_e} \right] = \frac{1}{2} e^{-\hat{\beta}[1 \mp \alpha]} \tag{43}$$

where $\hat{P}'_e \equiv \frac{1}{2} e^{-\hat{\beta}}$

For a given $\hat{P}'_e$, ϵ, α this equation can be solved for n. Choosing the sign yielding the largest n gives the following conservative result.

$$n \geq 2K_\epsilon^2 \, e^{\hat{\beta}} \left[1 - e^{-\hat{\beta}\alpha} \right]^{-2} \tag{44}$$

This result is plotted in Fig. 10 for values of α corresponding to errors in β of $\pm$ 0.2 db and $\pm$0.4 db.

From Eqs. (39), (40) and (42) the number of samples used for SNR ($\hat{\rho}$) measurements must be at least as large as the largest value of n satisfying

$$\hat{\rho}' \left[1 \pm \frac{K_\epsilon}{\sqrt{n}} \sqrt{\frac{1}{\hat{\rho}'} + \frac{n}{2(n-1)}} \right]^2 = \hat{\beta}(1 \pm \alpha) \left[1 + \frac{1}{2\hat{\beta}(1 \pm \alpha)} \right]^{-1} \tag{45}$$

where $\hat{\rho}' = \hat{\beta}\, [\, 1 + \frac{1}{2\hat{\beta}} \,]^{-1}$

Now, for $\beta > 2$

$$[\, 1 + \frac{1}{2\beta(1 \pm \alpha)} \,]^{-1} \simeq 1$$

Similarly for n > 1000

$$\left[1 \pm \frac{K_\epsilon}{\sqrt{n}} \sqrt{\frac{1}{\hat{\rho}'} + \frac{n}{2(n-1)}} \right]^2 \simeq \left[1 \pm \frac{K_\epsilon}{\sqrt{n}} \sqrt{\frac{1}{\hat{\rho}'} + \frac{1}{2}} \right]^2$$

Thus for $\beta > 2$ and n > 1000 Eq. (45) becomes approximately

$$\hat{\rho}' \left[1 \pm \frac{K_\epsilon}{\sqrt{n}} \sqrt{\frac{1}{\hat{\rho}'} + \frac{1}{2}} \right]^2 = \hat{\beta}(1 \pm \alpha)$$

Solving this for the largest value of n gives

$$n \geq K_\epsilon^2 \frac{[\frac{1}{\beta} + \frac{1}{2}\,]}{[1 - \sqrt{1-\alpha}^{\,2}]} \tag{46}$$

Fig. 10 illustrates this result for errors in $\hat{\beta}$ of $\pm$ 0.2 db and $\pm$ 0.4 db.

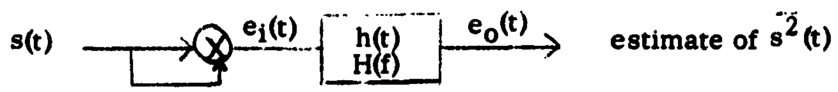
Heuristic reasoning suggests that due to the small noise variance at high SNR (and thus low P_e) it should be possible to measure $\hat{\beta}$ via SNR with relatively few samples as compared to measurement via P_e. Fig. 10 provides a striking confirmation of this conjecture and demonstrates that for $\beta > 2$ a given accuracy can always be obtained with

fewer samples via SNR measurements. Even more important, however, is the fact that for β on the order of 10 or greater (corresponding to $P_e < 10^{-5}$) the reduction in sample size is several orders of magnitude. This fact coupled with accurate knowledge of the measurement accuracy for a given sample size allows meaningful evaluation of system performance with an absolute minimum of samples.

III. CONTINUOUS ESTIMATION ERRORS

This section is concerned with the errors involved in the estimation of the mean square value of an ergodic Gaussian random process. The continuous and time-varying estimate considered here is obtained from a finite time average of the square of a sample function of the process. A true RMS voltmeter is an example of a device utilizing this method of measurement; the meter indicates the instantaneous output of a low pass filter that provides the weighted time average. Due to the finite averaging time (or equivalently the non-zero filter bandwidth) the estimate varies with time; the rate of variation being inversely proportional to the bandwidth of the process. The following analysis will obtain probabilistic bounds on the magnitude of these variations that are roughly analogous to the confidence interval relations for discrete estimates discussed in the previous section.

Consider the following circuit which provides the estimate described above.



From the Superposition Integral it follows that

$$e_o(t) = \int_{-\infty}^{\infty} s^2(\tau)\, h(t-\tau)\, d\tau$$

Thus the average value of $e_o(t)$ (either ensemble or infinite time) is

$$E[e_o(t)] \equiv \mu \qquad = E\left[\int_{-\infty}^{\infty} s^2(\tau)\, h(t-\tau)\, d\tau\right]$$

$$= E[s^2(\tau)]\ H(0) = R(0)\, H(0) \tag{47}$$

Assuming that H(t) is normalized (i.e. H(0) = 1) this demonstrates that the average value of the estimate is equal to the mean square value of the input signal. However, due to the finite filter bandwidth, $e_o(t)$ will vary about its mean value. These variations are most conveniently investigated by determining the variance of $e_o(t)$.

By definition

$$\mathrm{Var}[e_o(t)] = E[e_o^2(t)] - E[e_o(t)]^2 = \overline{e_o^2(t)} - R^2(0)$$

Since s(t) is assumed to be Gaussian it is known that [9]

$$\overline{e_o^2(t)} = \int_{-\infty}^{\infty} S_i(f)\, |H(f)|^2\, df$$

where

$$S_i(f) = R^2(0)\, \delta(f) + 2\int_{-\infty}^{\infty} S(f')\, S(f-f')\, df'$$

$\delta(f)$ = unit impulse function

$S(f)$ = power spectral density of s(t)

Thus

$$\mathrm{Var}[e_o(t)] \equiv \sigma^2 = 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S(f')\, S(f-f')\, |H(f)|^2\, df'\, df \tag{48}$$

or, equivalently

$$\sigma^2 = 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |H(f)|^2\, e^{-j2\pi f\tau}\, R^2(\tau)\, d\tau\, dt \tag{49}$$

where

$$R(\tau) = R(-\tau) = \int_{-\infty}^{\infty} S(f)\, e^{j2\pi f\tau}\, d\tau \tag{50}$$

Thinking in terms of the RMS voltmeter, a convenient method for using this variance to indicate the estimation error is to specify in db (relative to the average reading) the value corresponding to three standard deviations above the average reading. Let this estimation error or deviation parameter be denoted by D. Then from above

$$D = 10 \log_{10} \left[1 + \frac{3\sigma}{\mu} \right] \text{db} \tag{51}$$

For D to be small it is necessary that the filter bandwidth be narrow with respect to the signal bandwidth. However, under this condition the filter output is approximately Gaussian. Thus for small D the meter reading will be within $\pm$ D db of the correct value more than 99% of the time. Considered in this manner, D is roughly equivalent to the confidence interval for noise power discussed in the preceding section.

Due to the difficulty in evaluating Eq. (48) for arbitrary S(f) and H(f) it is desirable to provide an upper bound to Eq. (51) that does not involve the double integral of Eq. (48). This is readily accomplished as follows. The inner integration of Eq. (48) is the convolution of S(f) with itself. Let the resulting spectrum be $S_1(f)$. Then from the Schwartz inequality it follows that *

$$S_1(f) < S_1(0) \qquad f \neq 0$$

Thus σ^2 can be upper bounded by

$$\sigma^2 < 2S_1(0) \int_{-\infty}^{\infty} |H(f)|^2 df = 2 \int_{-\infty}^{\infty} S^2(f)\, df \int_{-\infty}^{\infty} |H(f')|^2\, df' \tag{52}$$

---

* By definition $S_1(f) = \int_{-\infty}^{\infty} S(f')\, S(f-f')\, df'$,

and

$$S_1(0) = \int_{-\infty}^{\infty} S^2(f)\, df' = \int_{-\infty}^{\infty} S^2(f-f')\, df'$$

Thus from the Schwartz inequality

$$\frac{\left[\int_{-\infty}^{\infty} S(f')\; S(f-f')\, df' \right]^2}{\int_{-\infty}^{\infty} S^2(f')\, df' \int_{-\infty}^{\infty} S^2(f-f')\, df'} \equiv \frac{S_1^2(f)}{S_1^2(0)} \leq 1$$

where the equality holds if and only if S(f') = S(f-f'), i.e. when f=0. It should be noted, however, that physical reasoning shows that Eq. (52) will introduce negligible error when the bandwidth of S(f) is much greater than that of the filter.

Since H(f) is assumed to be normalized the second integral in Eq. (52) defines the noise bandwidth W_n of the filter. In this discussion it is convenient to define the "effective integration time constant" of the filter by $T=(2W_n)^{-1}$. Then an upper bound for $(\sigma/\mu)^2$ can be written as

$$\frac{\sigma^2}{\mu^2} < \frac{1}{T} \frac{\int_{-\infty}^{\infty} S^2(f)\, df}{[\int_{-\infty}^{\infty} S(f)\, df\,]^2} \tag{53}$$

Define a signal " noise bandwidth" by

$$W_S = \int_{-\infty}^{\infty} \frac{S(f)}{\max S(f)}\, df \tag{54}$$

Then Eq. (53) can be written as

$$\frac{\sigma^2}{\mu^2} = \frac{T}{W_S^2} \int_{-\infty}^{\infty} \left[\frac{S(f)}{\max S(f)}\right]^2 df$$

Now, S(f) is by definition a non-negative function. Thus it follows that

$$\left[\frac{S(f)}{\max S(f)}\right]^2 \leq \frac{S(f)}{\max S(f)} \leq 1$$

and therefore that

$$\int_{-\infty}^{\infty} \left[\frac{S(f)}{\max S(f)}\right]^2 df \leq W_S \tag{55}$$

Substituting this result into Eq. (53) yields

$$\frac{\sigma^2}{\mu^2} < \frac{1}{T W_S} \tag{56}$$

and thus from Eq. (51)

$$D < 10 \log_{10} [\, 1+3(\frac{1}{TW_S})^{1/2}]\ \text{db} \tag{57}$$

with H(0) = 1, where W_S is defined by Eq. (54) and,

$$T = [\int_{-\infty}^{\infty} |H(f)|^2\, df\,]^{-1} \tag{58}$$

Eq. (57) is plotted as curve 1 in Figure 11 along an exact expression derived in the example below. These curves indicate that Eq. (57) provides a convenient and sufficiently accurate method for determining the required filter "integration time constant" Twhere a spectrum shape and a measurement accuracy are given.

Example 1

Let

$$H(f) = [1+j2\pi T_1 f]^{-1}$$

$$S(f) = \begin{cases} 1 & |f| < W \\ 0 & |f| > W \end{cases}$$

Consider first the exact calculations via Eqs. (47), (48) and (51).

$$\mu = \int_{-\infty}^{\infty} S(f)\,df = 2W$$

$$\sigma^2 = 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} S(f')\,S(f-f')\,|H(f)|^2\,df\,df'$$

$$= 4\int_0^{2W} \frac{[2W-f]}{1+4\pi^2T_1^2f^2}\,df$$

$$= \frac{4W}{\pi T_1}\,\mathrm{Tan}^{-1}[4\pi T_1W] - \frac{1}{2\pi^2T_1^2}\,\ln\,[1+16\pi^2T_1^2W^2]$$

$$D = 10\log_{10}[1+3\frac{\sigma}{\mu}\rho]$$

$$= 10\log_{10}\left\{1+3\,\frac{1}{\pi T_1W}\,\mathrm{Tan}^{-1}\left[4\pi T_1W - \frac{1}{2}\,(\frac{1}{2\pi T_1W})^2\ln\,[1+16\pi^2T_1^2W^2]\right]^{1/2}\right\} \tag{59}$$

For TW > 10 Eq. (59) becomes, with negligible error,

$$D = 10\log_{10}[1+3(\frac{1}{2T_1W})^{1/2}]\ \mathrm{db} \tag{60}$$

The upper bound of Eq. (57) is evaluated for this example as

$$D \leq 10 \log_{10} [1+3(\frac{1}{TW_s})^{1/2}] \text{ db} \tag{61}$$

where

$$T = [2 \int_{-\infty}^{\infty} |H(f)|^2 df]^{-1}$$

$$= [2 \int_{-\infty}^{\infty} \frac{1}{1+4\pi^2 T_1^2 f^2} df]^{-1}$$

$$= T_1$$

$$W_S = \int_{-\infty}^{\infty} S(f)\, df = 2W$$

Thus the upper baud of Eq. (61) and the asymptotic expression of Eq. (60) are identical when S(f) is a rectangular spectrum. This is in complete agreement with the approximations used in deriving Eq. (58).

Example II

Let

$$S(f) = [1+(\pi f/2W)^2]^{-1}$$

$$H(f) = [1+j\, 2\pi T_1 f]^{-1}$$

The exact analysis gives

$$\mu = \int_{-\infty}^{\infty} S(f)\, df = 2W$$

$$R(\tau) = \int_{-\infty}^{\infty} S(f)\, e^{j2\pi f T} df = 2W\, e^{-4W|\tau|}$$

From Eq. (49) the variance is

$$\sigma^2 = 2 \int_{-\infty}^{\infty} R_H(\tau)\, R^2(\tau)\, d\tau$$

where

$$R_H(\tau) = \int_{-\infty}^{\infty} |H(f)|^2 e^{-j\pi f\tau} df$$

For this H(f)

$$R_H(\tau) = \frac{1}{2T} e^{-\left|\frac{\tau}{T}\right|}$$

Thus

$$\sigma^2 = 8 \frac{W^2}{T} \int_{-\infty}^{\infty} \exp - [8W + \frac{1}{T}] \tau \, d\tau$$

$$= \frac{8W^2}{8WT+1}$$

and

$$D = 10 \log_{10} \left\{1+3 \left[\frac{2}{8WT+1}\right]^{1/2}\right\} \text{ db} \tag{62}$$

For TW > 10 Eq. (62) becomes, with negligible error

$$D = 10 \log_{10} [1+3\sqrt{\frac{1}{4WT}}] \text{ db} \tag{63}$$

Evaluating T and W_S for the upper bound gives $T = T_1$ and $W_S = 2W$. Thus the asymptotic expression, Eq. (63), differs from the upper bound by a factor of $\frac{1}{2}$ under the radical. For large TW_S this results in a negligible difference between the respective values of D. Eq. (62) is plotted vs TW_S as curve 2 in Fig. 11.

IV. SUMMARY

In this report the statistical concept of confidence intervals has been applied to the problem of obtaining probabilistic bounds on the errors arising when a finite number of samples are used to estimate the following parameters of a Gaussian random process.

a) the squared mean--called signal power,

b) the variance--called noise power,

c) the ratio of squared mean to variance--called SNR,

d) the probability that the sample value exceeds an arbitrary threshold--called P_e.

These results are useful in evaluating digital communication systems by providing information as the accuracy of test measurements and by allowing tests to be designed to provide the required accuracy with a minimum of data. An example has been given in which the use of these results allows a reduction, by several orders of magnitude, in the amount of data required to obtain a specified accuracy.

A bound roughly analogous to the confidence interval for the discrete estimate has been obtained for continuous estimates (e.g. an RMS voltmeter) of noise power. The bound is a function only of the noise bandwidth of the averaging filter and of the signal and is shown to be quite accurate when the measurement accuracy is reasonably small (< 0.5 db).

CAPTIONS

Fig. 1. 95% Confidence Intervals for Probability of Error

Fig. 2. 99% Confidence Intervals for Probability of Error

Fig. 3. Confidence Intervals for Noise Power

Fig. 4. 95% Confidence Intervals for Signal Power

Fig. 5. 99% Confidence Intervals for Signal Power

Fig. 6. 95% Confidence Intervals for SNR

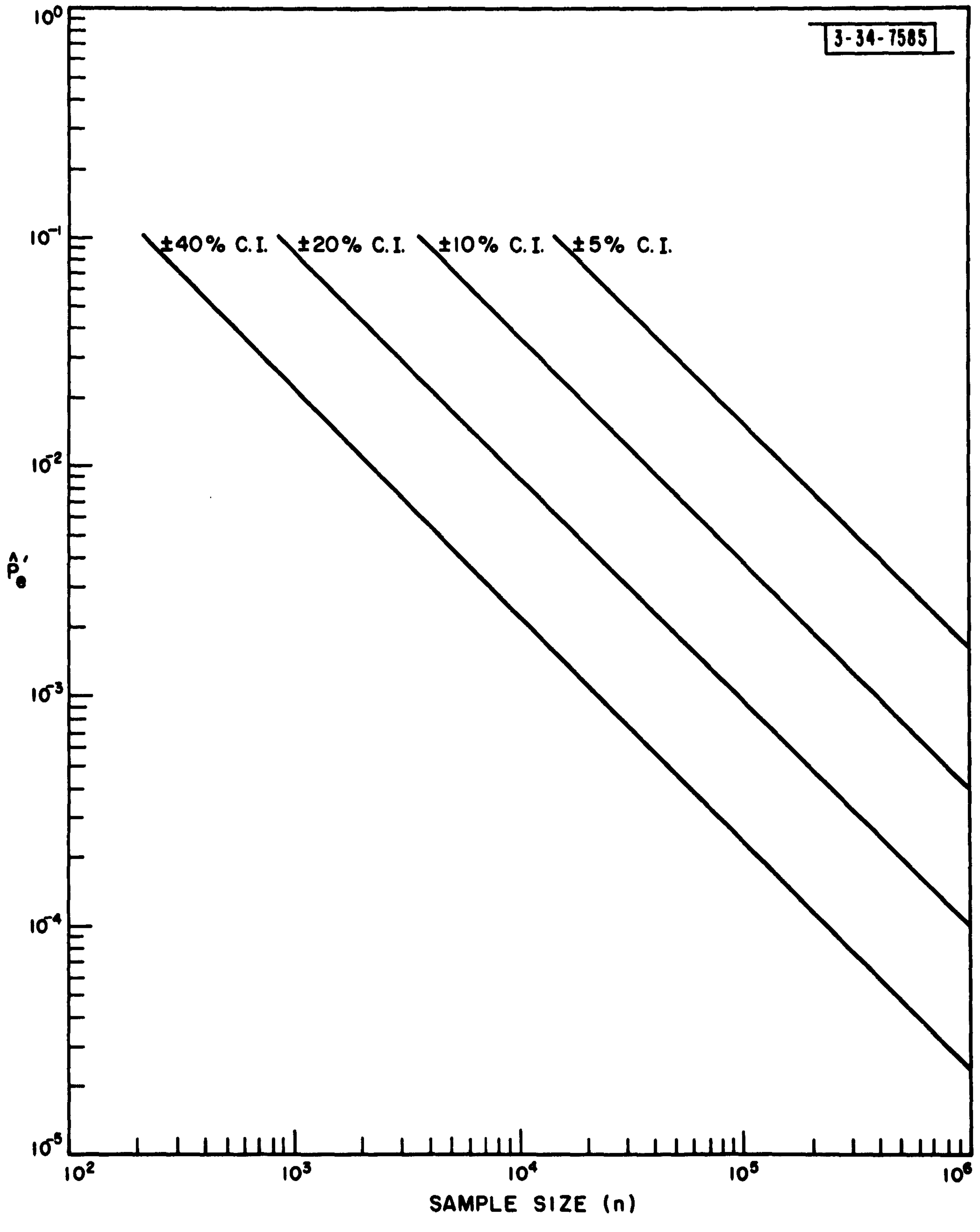
Fig. 7. 99% Confidence Intervals for SNR

Fig. 8. 95% Confidence Intervals for SNR

Fig. 9. 99% Confidence Intervals for SNR

Fig. 10. Comparison of Measurements via P_e and SNR

Fig. 11. Estimation Error for the Continuous Estimate of the Mean Square Value of a Random Process.



3-34-7585
±40% C.I.
±20% C.I.
±10% C.I.
±5% C.I.
$\hat{P}_e'$
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
10^2
10^3
10^4
10^5
10^6
SAMPLE SIZE (n)

FIGURE 1

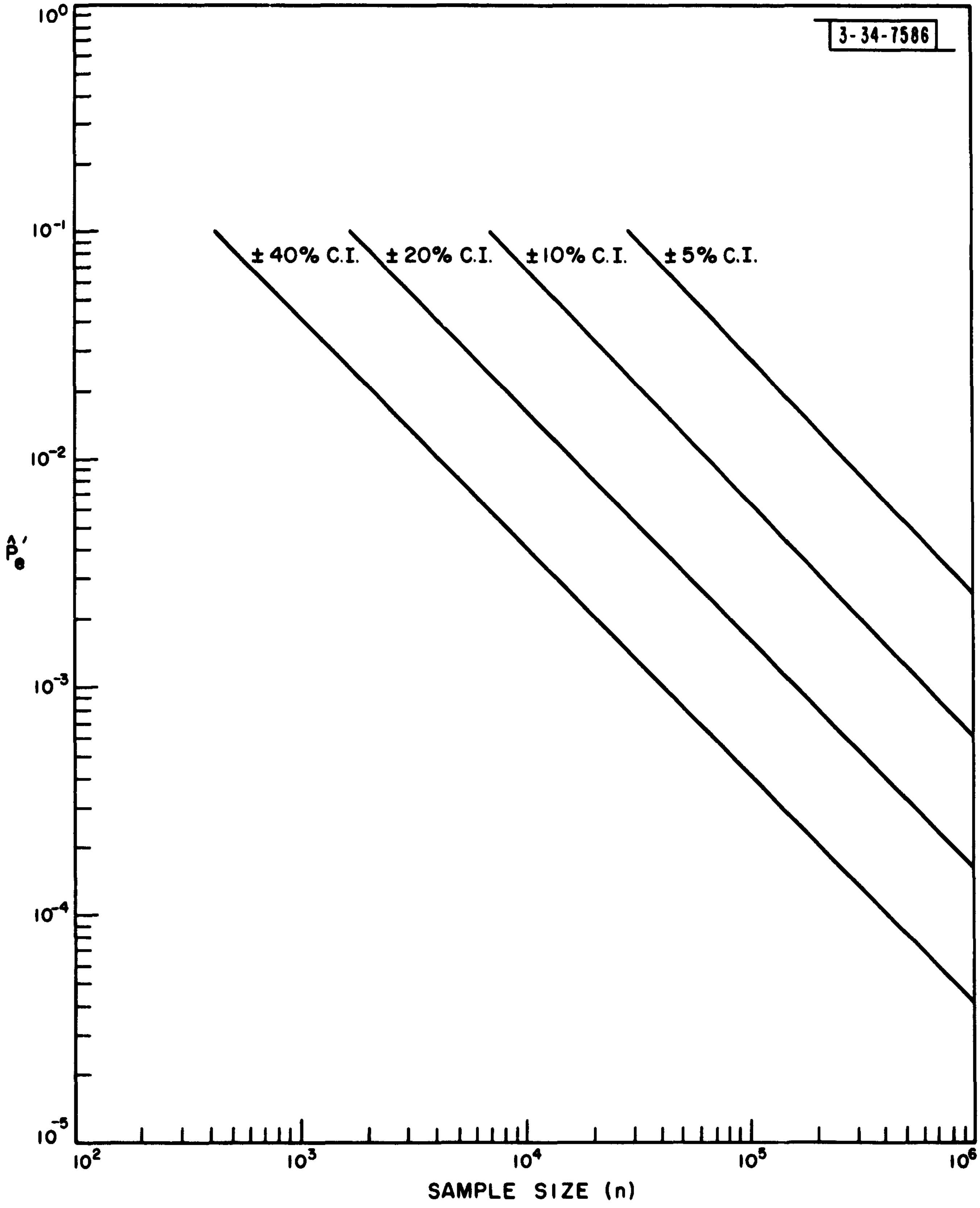

3-34-7586
± 40% C.I.
± 20% C.I.
± 10% C.I.
± 5% C.I.
10^0
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-5}
$\hat{P}_e'$
10^2
10^3
10^4
10^5
10^6
SAMPLE SIZE (n)

FIGURE 2

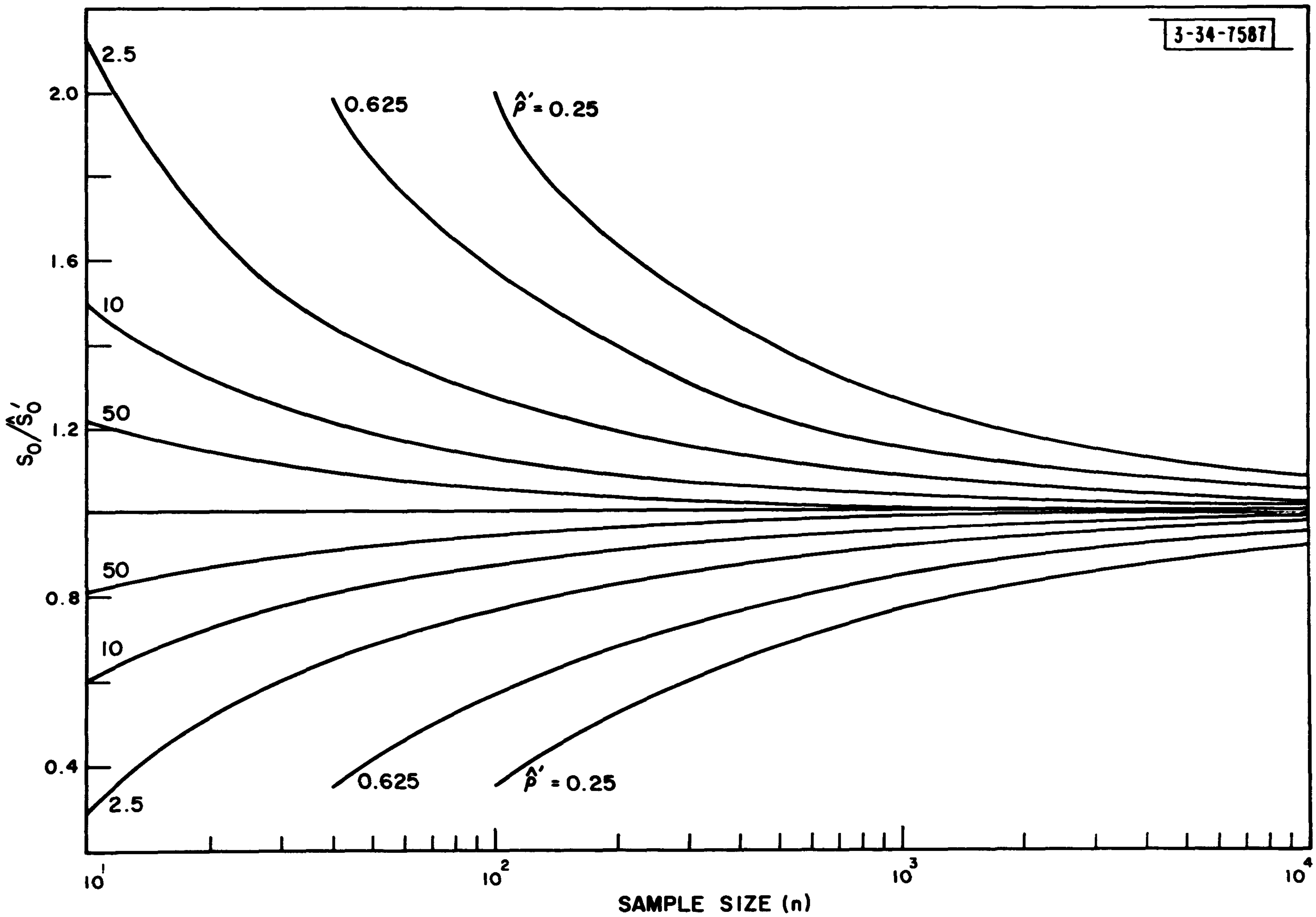

3-34-7587
2.5
0.625
$\hat{\rho}'$ = 0.25
10
50
50
10
2.5
0.625
$\hat{\rho}'$ = 0.25
2.0
1.6
1.2
0.8
0.4
$S_0/\hat{S}_0'$
10^1
10^2
10^3
10^4
SAMPLE SIZE (n)

FIGURE 3

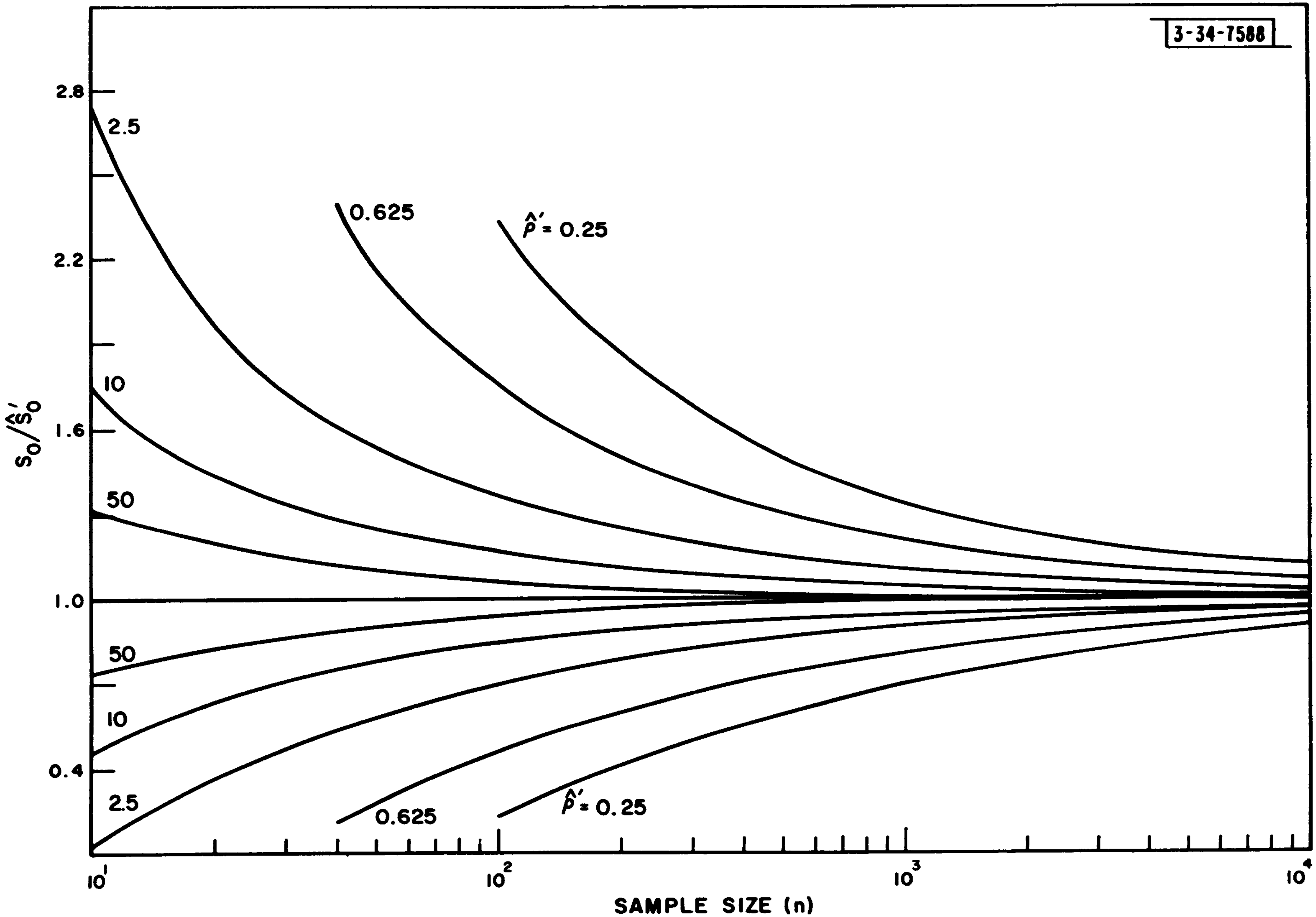

3-34-7588
2.8
2.2
1.6
1.0
0.4
$S_0/\hat{S}_0'$
2.5
0.625
$\hat{\rho}' = 0.25$
10
50
50
10
2.5
0.625
$\hat{\rho}' = 0.25$
10^1
10^2
10^3
10^4
SAMPLE SIZE (n)

FIGURE 4

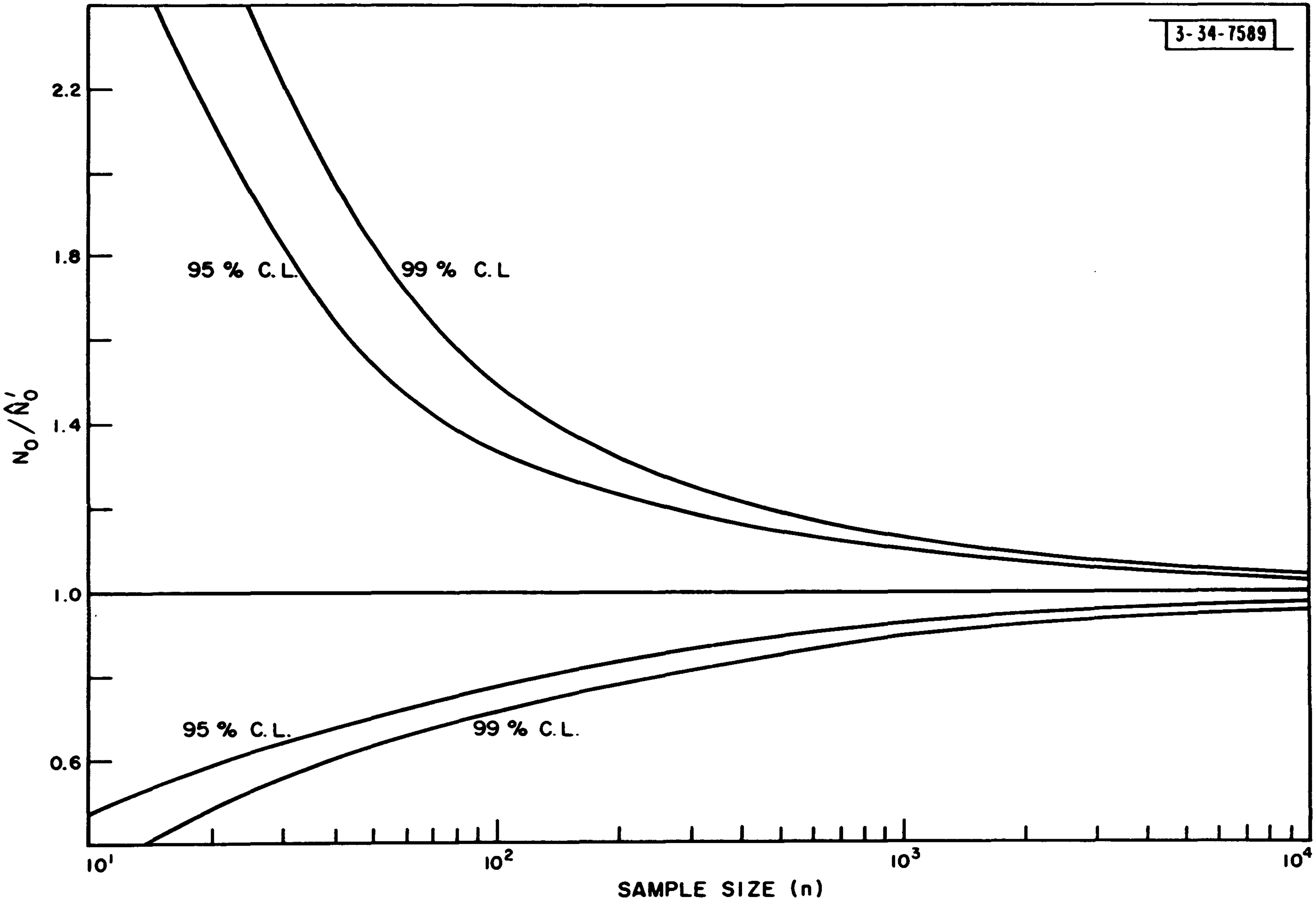
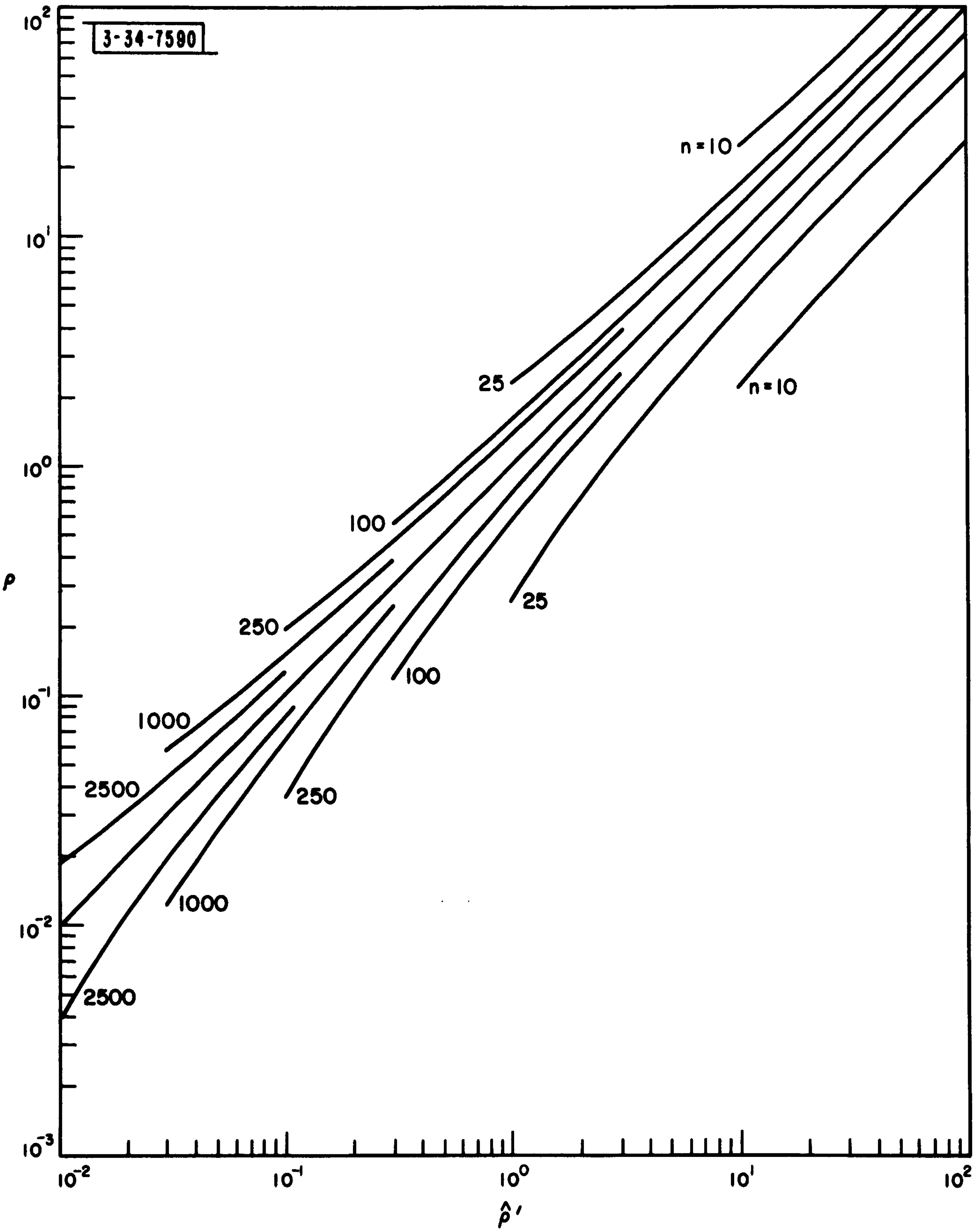

3-34-7589
2.2
1.8
1.4
1.0
0.6
$N_0/\hat{N}_0'$
95 % C.L.
99 % C.L
95 % C.L.
99 % C.L.
10^1
10^2
10^3
10^4
SAMPLE SIZE (n)

FIGURE 5



3-34-7590
n=10
25
100
250
1000
2500
n=10
25
100
250
1000
2500
ρ
$\hat{\rho}'$
10^{2}
10^{1}
10^{0}
10^{-1}
10^{-2}
10^{-3}
10^{-2}
10^{-1}
10^{0}
10^{1}
10^{2}

FIGURE 6

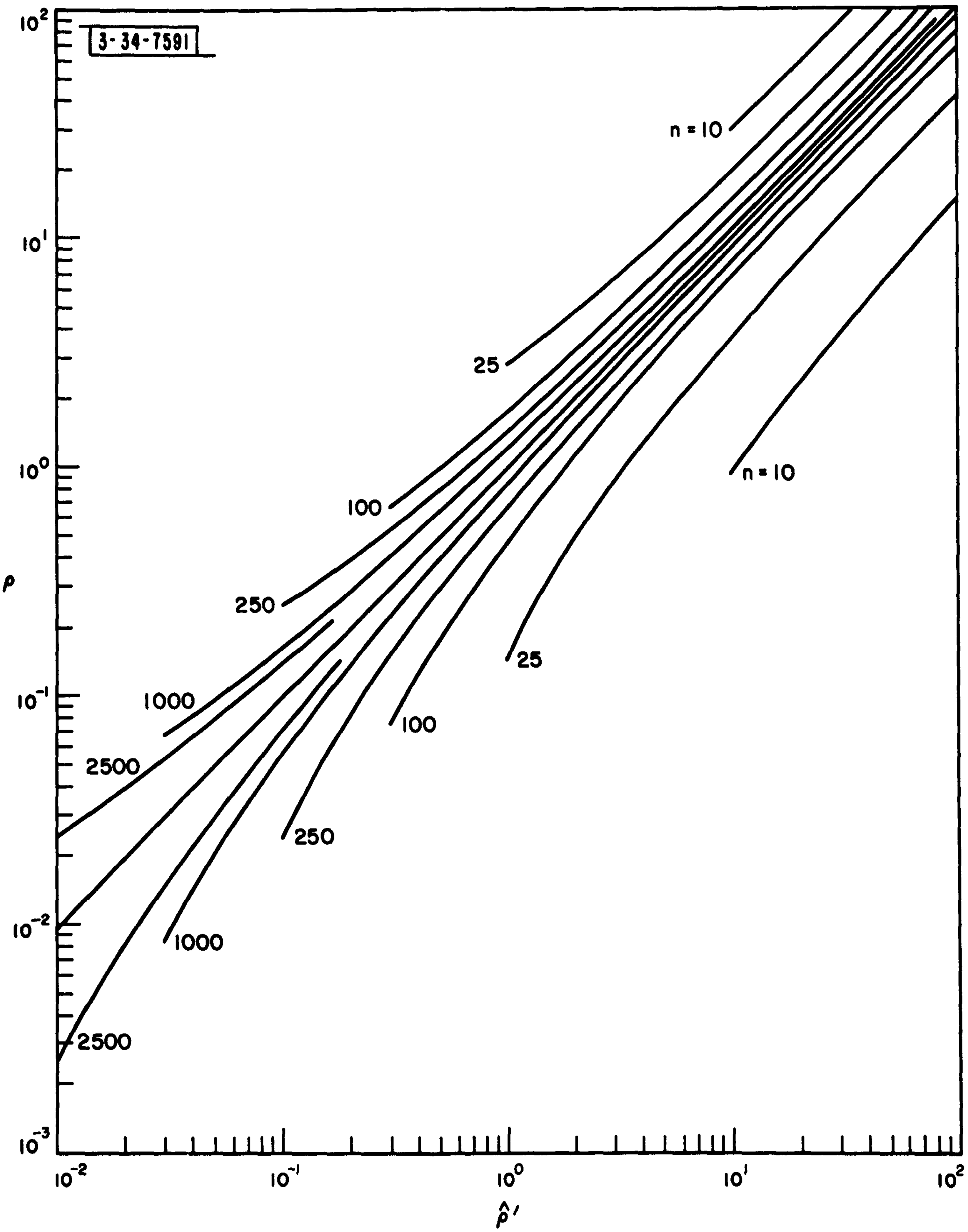
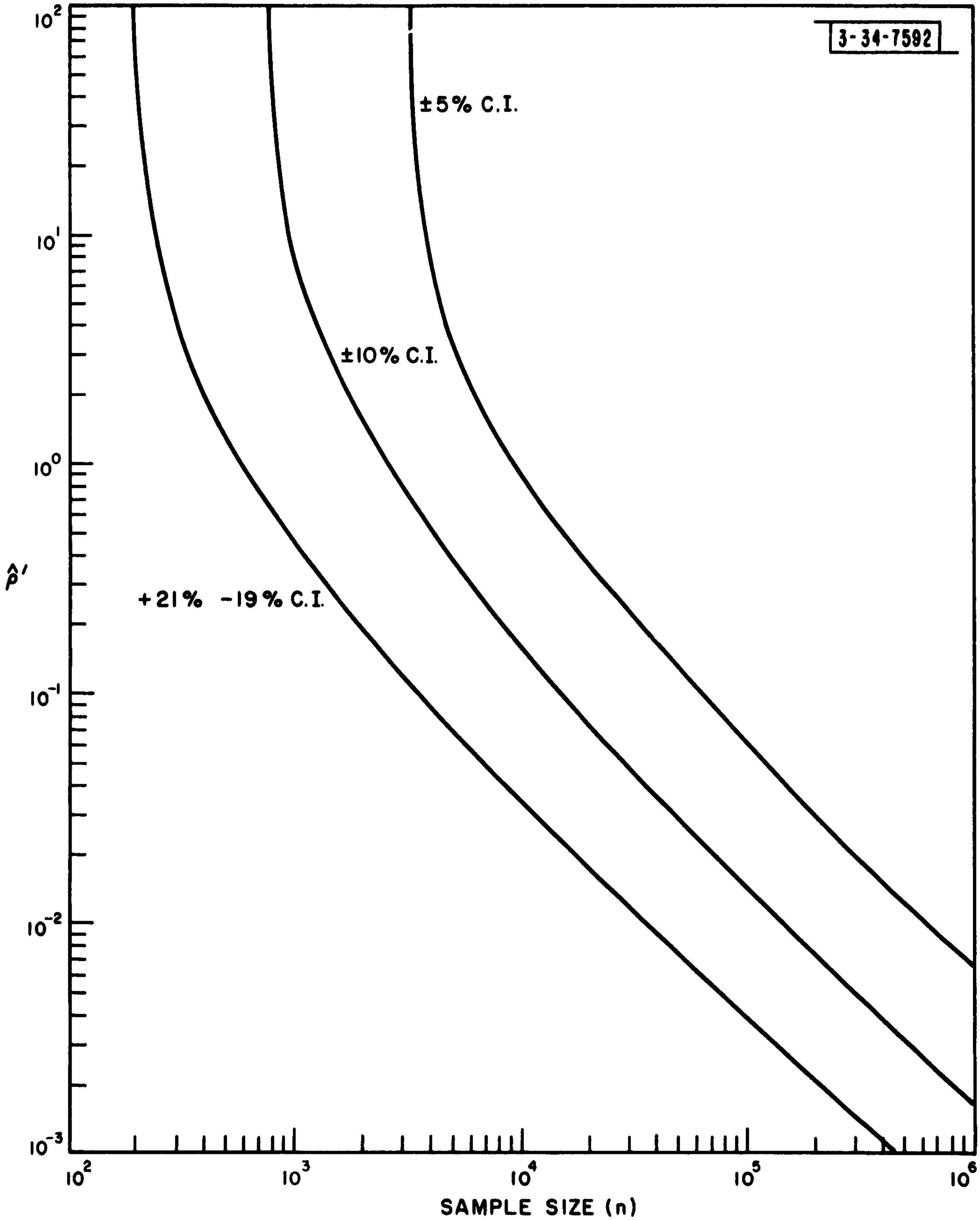


FIGURE 7



3-34-7592
±5% C.I.
±10% C.I.
+21% -19% C.I.
$\hat{\rho}'$
10^2
10^1
10^0
10^{-1}
10^{-2}
10^{-3}
10^2
10^3
10^4
10^5
10^6
SAMPLE SIZE (n)

FIGURE 8

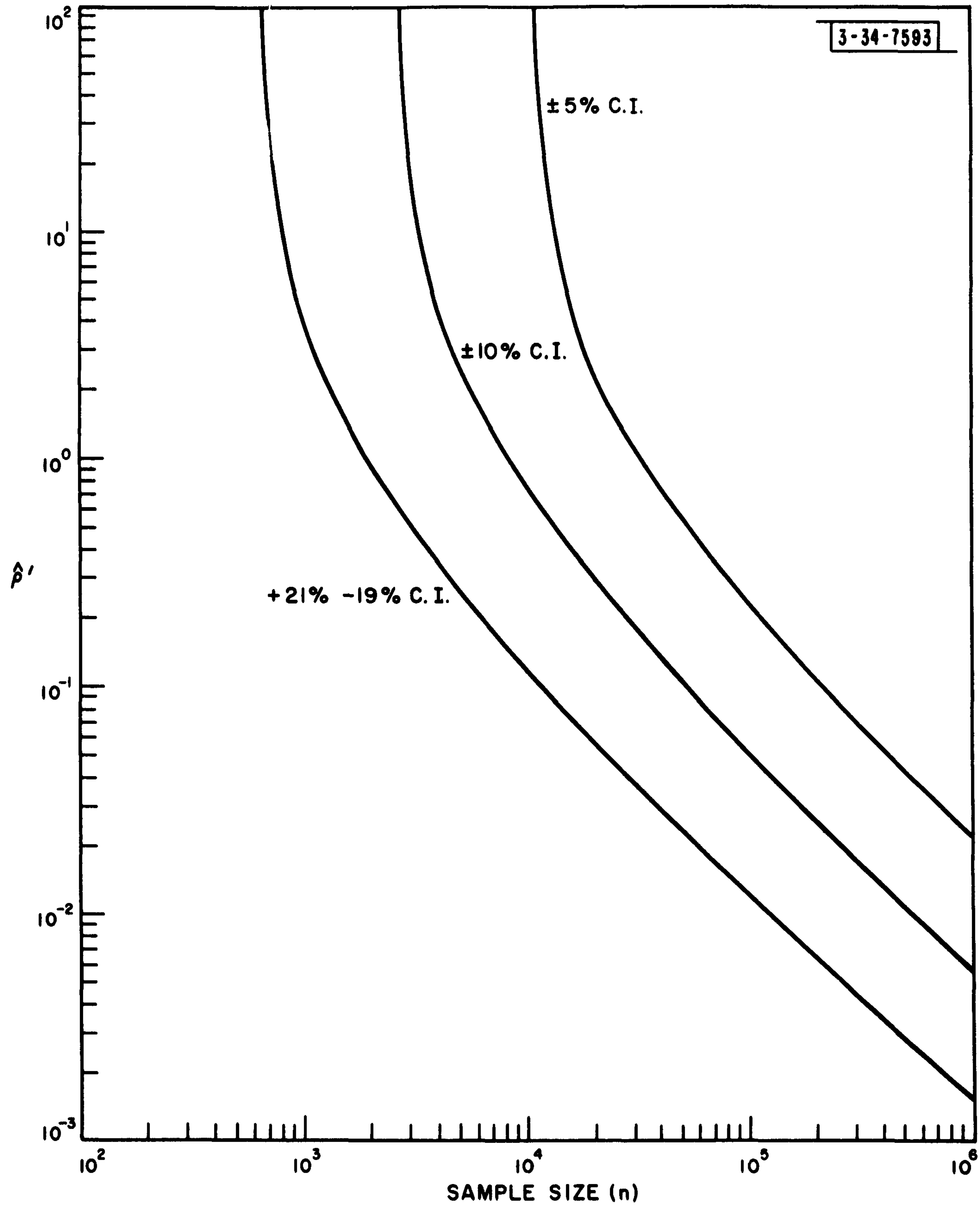
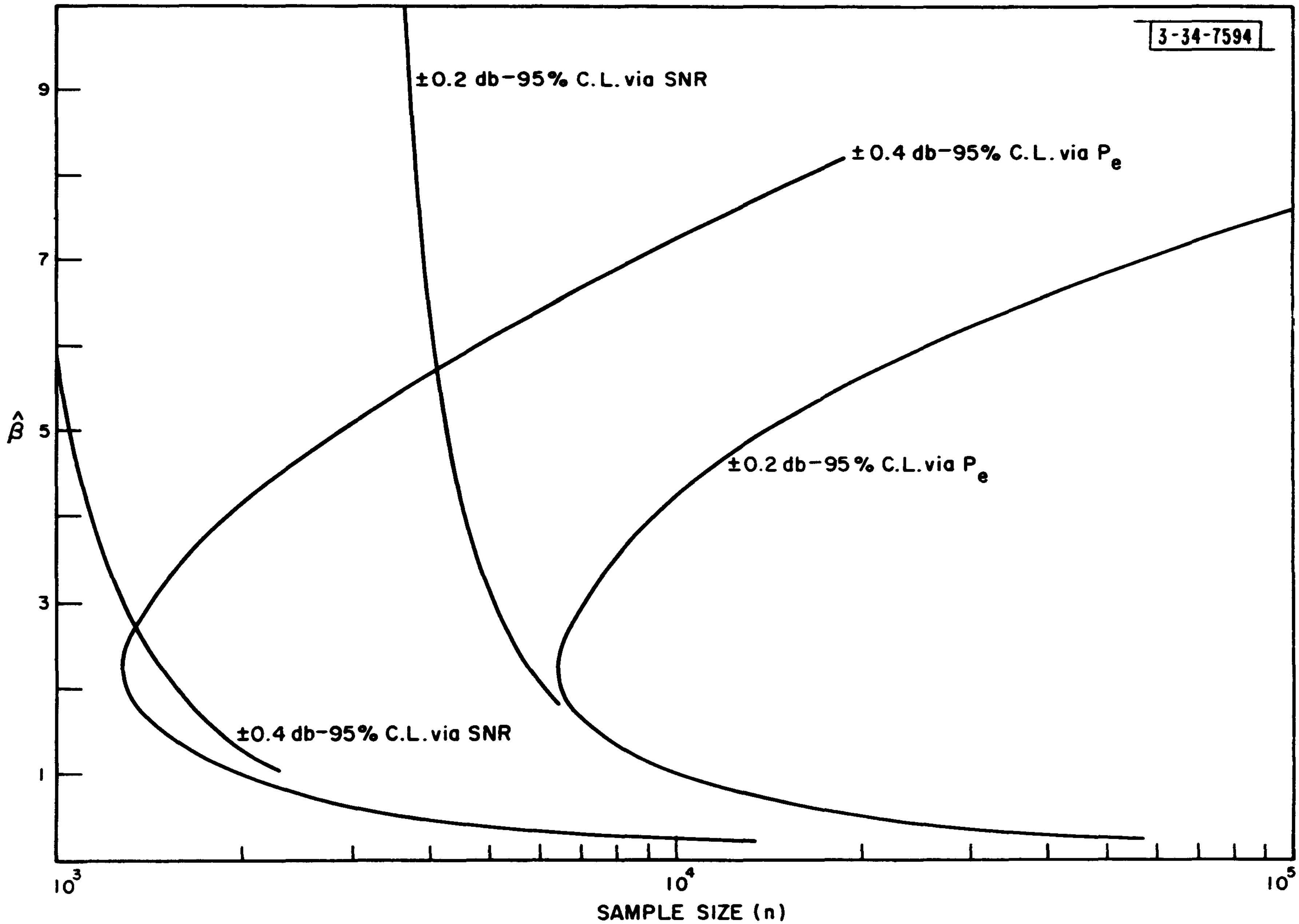

3-34-7593
± 5% C.I.
± 10% C.I.
+ 21% −19% C.I.
$\hat{p}'$
10^2
10^1
10^0
10^{-1}
10^{-2}
10^{-3}
10^2
10^3
10^4
10^5
10^6
SAMPLE SIZE (n)

FIGURE 9



3-34-7594
±0.2 db−95% C.L. via SNR
±0.4 db−95% C.L. via P_e
±0.2 db−95% C.L. via P_e
±0.4 db−95% C.L. via SNR
$\hat{\beta}$
9
7
5
3
1
10^3
10^4
10^5
SAMPLE SIZE (n)

FIGURE 10

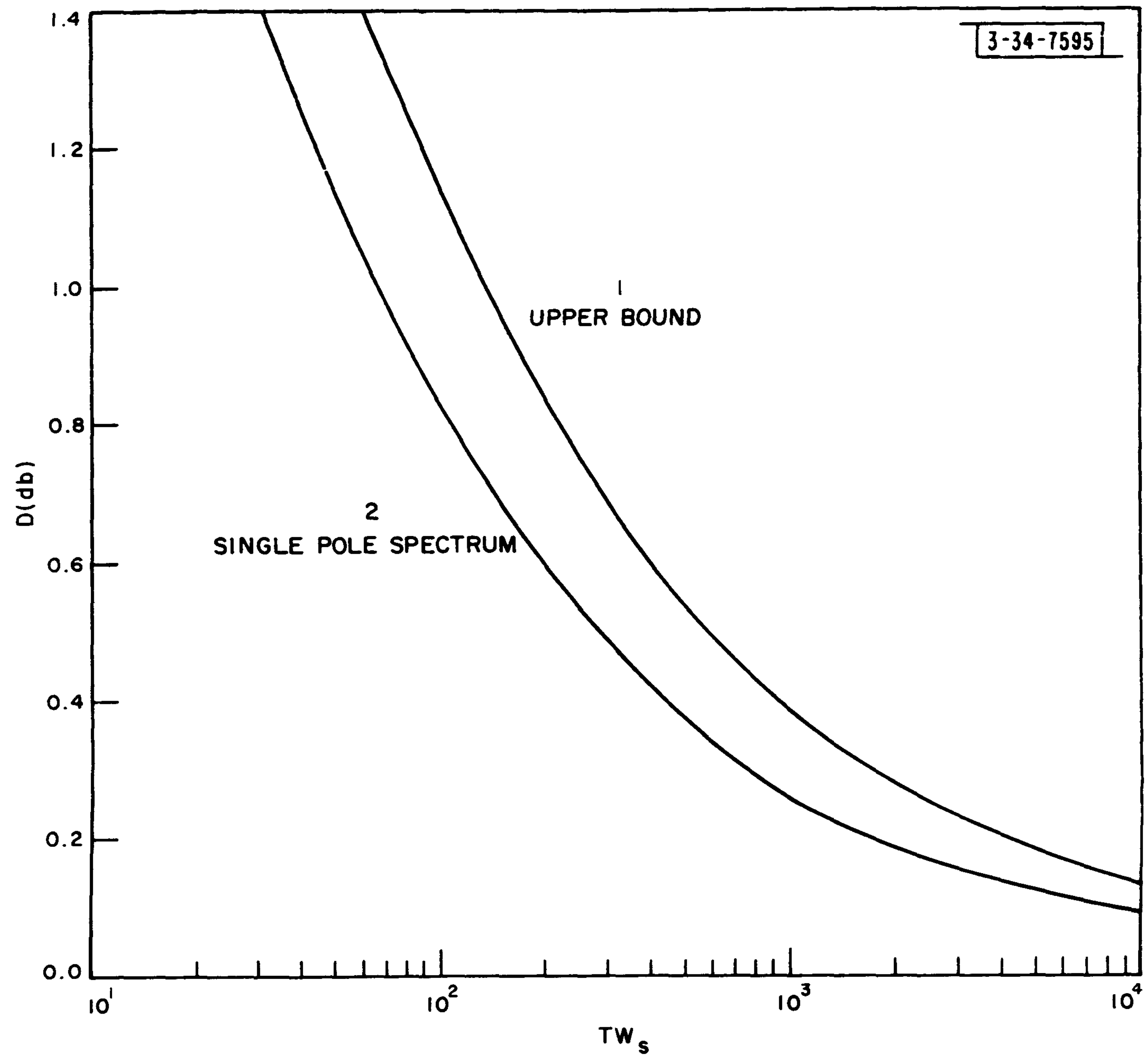

3-34-7595
1
UPPER BOUND
2
SINGLE POLE SPECTRUM
D(db)
1.4
1.2
1.0
0.8
0.6
0.4
0.2
0.0
10^1
10^2
10^3
10^4
TW_s

FIGURE 11

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